

REPRESENTATIONS ADMITTING TWO PAIRS OF SUPPLEMENTARY INVARIANT SPACES

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ABSTRACT. We examine the lattice generated by two pairs of supplementary vector subspaces of a finite-dimensional vector-space by intersection and sum, with the aim of applying the results to the study of representations admitting two pairs of supplementary invariant spaces, or one pair and a reflexive form. We show that such a representation is a direct sum of three canonical sub-representations which we characterize. We then focus on holonomy representations with the same property.

1. INTRODUCTION

A famous paper of Gelfand and Ponomarev [GP] classifies the systems on four vector subspaces of a finite-dimensional vector space. We focus on the systems of two pairs of supplementary spaces and explore the lattice generated by sum and intersection starting from the four spaces. The aim is to apply the results to lattices of stable spaces of finite-dimensional representations and in particular of holonomy representations of torsion free connections preserving a reflexive form.

2. LATTICE GENERATED BY TWO PAIRS OF SUPPLEMENTARY SPACES

We suppose throughout the paper that \mathbb{K} is a commutative field of characteristic different from 2.

2.1. Definitions. We call *decomposition of a finite-dimensional \mathbb{K} -vector space E into 2 direct sums* a quintuplet $\mathcal{V} = (E, V_1, V_2, W_1, W_2)$ where V_1, V_2, W_1 and W_2 are four vector subspaces of the finite-dimensional vector space E verifying $V_1 \oplus V_2 = W_1 \oplus W_2 = E$.

Example 1. *In particular if E carries a non-degenerate reflexive structure (i.e. for us a non-degenerate symmetric or antisymmetric bilinear form) and if $E = V_1 \oplus V_2$ then $(E, V_1, V_2, V_1^\perp, V_2^\perp)$ is a decomposition of E into 2 direct sums.*

Associated to a decomposition of a finite-dimensional \mathbb{K} -vector space E into 2 direct sums $\mathcal{V} = (E, V_1, V_2, W_1, W_2)$ is a dual decomposition into two direct sums: $\mathcal{V}^* = (E^*, W_1', W_2', V_1', V_2')$, with $X' := \{u \in E^* \mid u(X) = 0\}$.

If $E = E_1 \oplus E_2$ is a direct sum, let $p_{E_1}^{E_2}$ be the projection on E_1 parallelly to E_2 . To simplify notations let's write p_i for the projection on V_i parallelly

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to $V_{\tau(i)}$ and q_i the projection on W_i parallelly to $W_{\tau(i)}$. We define the map $\theta_{\mathcal{V}} : E \rightarrow E$ by $\theta_{\mathcal{V}} = p_{W_1}^{W_2} \circ p_{V_1}^{V_2} - p_{V_1}^{V_2} \circ p_{W_1}^{W_2}$. To simplify notations we write θ for $\theta_{\mathcal{V}}$ if it is clear which \mathcal{V} we mean.

It is easy to verify:

Lemma 1. $\theta = p_{W_1}^{W_2} \circ p_{V_1}^{V_2} - p_{V_1}^{V_2} \circ p_{W_1}^{W_2} = p_{W_2}^{W_1} \circ p_{V_2}^{V_1} - p_{V_2}^{V_1} \circ p_{W_2}^{W_1} = p_{V_2}^{V_1} \circ p_{W_1}^{W_2} - p_{W_1}^{W_2} \circ p_{V_2}^{V_1} = p_{V_1}^{V_2} \circ p_{W_2}^{W_1} - p_{W_2}^{W_1} \circ p_{V_1}^{V_2}$

We have also:

Lemma 2. $\theta(V_i) \subset V_{\tau(i)}$ and $\theta(W_i) \subset W_{\tau(i)}$

Lemma 3. If \mathcal{V}^* is the dual system of \mathcal{V} then $\theta_{\mathcal{V}^*} = (\theta_{\mathcal{V}})^*$

Proof. We have:

$$\begin{aligned} (\theta_{\mathcal{V}})^* &= (p_{W_1}^{W_2} \circ p_{V_1}^{V_2} - p_{V_1}^{V_2} \circ p_{W_1}^{W_2})^* \\ &= (p_{W_1}^{W_2} \circ p_{V_1}^{V_2})^* - (p_{V_1}^{V_2} \circ p_{W_1}^{W_2})^* \\ &= (p_{V_1}^{V_2})^* \circ (p_{W_1}^{W_2})^* - (p_{W_1}^{W_2})^* \circ (p_{V_1}^{V_2})^* \\ &= p_{V_2'}^{V_1'} \circ p_{W_2'}^{W_1'} - p_{W_2'}^{W_1'} \circ p_{V_2'}^{V_1'} \\ &= \theta_{\mathcal{V}^*} \end{aligned}$$

□

2.2. Canonical decomposition of E .

Definition 1. Let us define a sequence of vector subspaces of E : $F(0) := \{0\}$, $F(n+1) := \sum_{i,j} ((F(n) + V_i) \cap (F(n) + W_j))$ for $n \geq 0$.

$(F(n))_n$ is an increasing sequence of vector subspaces of the finite-dimensional vector space E and necessarily stationary. Let us write F or $F(\infty)$ the space $\sum_n F(n)$. F is the smallest fix-point of the increasing mapping $X \mapsto \sum_{i,j} ((X + V_i) \cap (X + W_j))$, and F is the smallest common fix-point of the four increasing mappings $X \mapsto (X + V_i) \cap (X + W_j)$ for $i, j \in \{1, 2\}$.

Lemma 4. $F(1) = \bigoplus_{i,j} V_i \cap W_j$

Proof. By definition we have $F(1) = \sum_{i,j} V_i \cap W_j$, and it is easy to see that the sum is necessarily direct. □

Definition 2. Let us define a sequence of vector subspaces of E : $\tilde{F}(0) := E$, $\tilde{F}(n+1) := \bigcap_{i,j} ((\tilde{F}(n) \cap V_i) + (\tilde{F}(n) \cap W_j))$ for $n \geq 0$.

$(\tilde{F}(n))_n$ is a decreasing sequence of vector subspaces of the finite-dimensional vector-space E and so stationary. Let $\tilde{F}(\infty)$ or simply \tilde{F} be the space $\bigcap_n \tilde{F}(n)$. \tilde{F} is the biggest fix-point of the decreasing mapping $X \mapsto \bigcap_{i,j} ((X \cap V_i) + (X \cap W_j))$, and \tilde{F} is the biggest common fix-point of the four decreasing mappings $X \mapsto (X \cap V_i) + (X \cap W_j)$, for $i, j \in \{1, 2\}$.

Proposition 5. For every non-negative integer n

- (1) $\ker \theta^n = F(n)$
- (2) $\text{im} \theta^n = \tilde{F}(n)$

Proof. (1) Let us show first that $\ker \theta = F(1) = V_1 \cap W_1 + V_1 \cap W_2 + V_2 \cap W_1 + V_2 \cap W_2$. If $x \in V_i \cap W_j$, $\theta(x) = (-1)^{i+j}((p_i \circ q_j)(x) - (q_j \circ p_i)(x)) = (-1)^{i+j}(x - x) = 0$. As θ is linear, $\theta(F(1)) = 0$.

Inversely if $\theta(x) = 0$, we have $(q_1 \circ p_1 - p_1 \circ q_1)(x) = 0$ and so $(q_1 \circ p_1)(x) = (p_1 \circ q_1)(x)$. We have $(q_1 \circ p_1)(x) \in V_1 \cap W_1$. Similarly $(q_j \circ p_i)(x) = (p_i \circ q_j)(x)$ and so $(q_j \circ p_i)(x) \in V_i \cap W_j$. We deduce $x = q_1(x) + q_2(x) = \sum_{i,j} (q_j \circ p_i)(x) \in \sum_{i,j} V_i \cap W_j = F(1)$.

Let us show $F(n) \subset \ker \theta^n$. For $n = 0$ it is clear. If $n = k + 1$, suppose $\ker \theta^k = F(k)$. Let $x \in F(n) = \sum_{i,j} ((F(k) + V_i) \cap (F(k) + W_j))$. x can be written $x_{11} + x_{22} + x_{12} + x_{21}$ with $x_{ij} \in ((F(k) + V_i) \cap (F(k) + W_j))$. $x_{ij} = y_{ij} + z_{ij} = t_{ij} + u_{ij}$ with $y_{ij}, t_{ij} \in F(k)$, $z_{ij} \in V_i$ and $u_{ij} \in W_j$. We have by induction hypothesis $\theta^k(y_{ij}) = 0$ and $\theta^k(t_{ij}) = 0$. By iterated application of lemma 2 we have $\theta^k(z_{ij}) \in V_{\tau^k(i)}$ et $\theta^k(u_{ij}) \in W_{\tau^k(j)}$. As a consequence $\theta^k(x_{ij}) \in V_{\tau^k(i)} \cap W_{\tau^k(j)}$ and so $\theta^k(x) \in F(1) = \ker \theta$, giving: $\theta^{k+1}(x) = 0$.

Let us show $\ker \theta^n \subset F(n)$. For $n = 0$, $\ker \theta^0 = \{0\} = F(0)$. For $n = k + 1$, suppose $\ker \theta^k \subset F(k)$. Let x be such that $\theta^n(x) = 0$. We have then $\theta^k(\theta(x)) = 0$. By induction hypothesis $\theta(x) \in F(k)$. So $(q_j \circ p_i)(x) - (p_i \circ q_j)(x) \in F(k)$ and as a consequence: $(q_j \circ p_i)(x) \in (F(k) + V_i)$. As $(q_j \circ p_i)(x) \in W_j$, $(q_j \circ p_i)(x) \in (F(k) + V_i) \cap W_j \subset (F(k) + V_i) \cap (F(k) + W_j)$. Finally $x = \sum_{i,j} (q_j \circ p_i)(x) \in \sum_{i,j} ((F(k) + V_i) \cap (F(k) + W_j)) = F(n)$.

(2) To show that $\text{im} \theta^n = \tilde{F}(n)$, we will use duality¹:

In finite dimension it is easy to show by induction that for every n , $(F_V(n))' = \tilde{F}_{V^*}(n)$ and $(\tilde{F}_V(n))' = F_{V^*}(n)$.

So we have: $(\tilde{F}_V(n))'' = (F_{V^*}(n))' = (\ker \theta_{V^*}^n)' = (\ker(\theta_V^*)^n)' = (\ker(\theta_V^n)^*)' = (\text{im} \theta_V^n)''$. By injectivity in finite dimension of " we have $\text{im} \theta_V^n = \tilde{F}_V(n)$.

□

Proposition 6. (1) $\forall n, F(n+1) = \theta^{-1}(F(n))$,
(2) $\forall n, \tilde{F}(n+1) = \theta(\tilde{F}(n))$.

Proof. We have: $F(n+1) = \ker(\theta^{n+1}) = \theta^{-1}(\ker(\theta^n)) = \theta^{-1}(F(n))$ et $\theta(\tilde{F}(n)) = \theta(\text{im}(\theta^n)) = \theta(\tilde{F}(n))$. □

From the first point one can deduce: $\forall n, \theta(F(n+1)) \subset F(n)$.

We recall without proof the following well known result:

Proposition 7. If E is a finite-dimensional vector space and Ψ an endomorphism of E then the two subspaces of E : $E_N = \sum_n \ker(\Psi^n)$ and $E_I = \bigcap_n \text{im}(\Psi^n)$ are stable by Ψ and we have $E = E_N \oplus E_I$. Moreover $\Psi|_{E_N}$ is nilpotent and $\Psi|_{E_I}$ is invertible.

The result applied to E and the endomorphism θ gives us for $F := \sum_n F(n)$ and $\tilde{F} := \bigcap_n \tilde{F}(n)$: $E = F \oplus \tilde{F}$. Moreover F and \tilde{F} are stable by θ and $\theta|_F$ is nilpotent and $\theta|_{\tilde{F}}$ is invertible.

¹We use the following lemma which is easy to show: For $\Psi \in \mathcal{L}(E, F)$, $\ker \Psi^* = (\text{im} \Psi)'$ and $\text{im} \Psi^* = (\ker \Psi)'$.

We say that the subspace V of E is *homogeneous* with respect to the sum $E_1 + E_2$, where E_1 and E_2 are vector subspaces of E if: $V \cap (E_1 + E_2) = (V \cap E_1) + (V \cap E_2)$. Similarly we say that V is *co-homogeneous* with respect to the intersection $E_1 \cap E_2$, if: $V + (E_1 \cap E_2) = (V + E_1) \cap (V + E_2)$.

Proposition 8. (1) $(\tilde{F} \cap V_1) \oplus (\tilde{F} \cap V_2) = \tilde{F}$
 (2) $(\tilde{F} \cap W_1) \oplus (\tilde{F} \cap W_2) = \tilde{F}$
 (3) $\forall i, j, (\tilde{F} \cap V_i) \oplus (\tilde{F} \cap W_j) = \tilde{F}$

Proof. Let us start by the proof of point 3. We have: $V_i \cap W_j \subset F(1)$, which gives us $(\tilde{F} \cap V_i) \cap (\tilde{F} \cap W_j) \subset \tilde{F} \cap F(1) = \{0\}$. From $\tilde{F} = (\tilde{F} \cap V_i) + (\tilde{F} \cap W_j)$ we deduce then $\tilde{F} = (\tilde{F} \cap V_i) \oplus (\tilde{F} \cap W_j)$.

Let us note $n_i = \dim(\tilde{F} \cap V_i)$ and $m_j := \dim(\tilde{F} \cap W_j)$. Point 3 implies then that $n_i + m_j = \dim \tilde{F}$ (*). This gives us $n_1 = n_2$ and $m_1 = m_2$.

As $V_1 \cap V_2 = \{0\}$, $(\tilde{F} \cap V_1) \cap (\tilde{F} \cap V_2) = \{0\}$. As $(\tilde{F} \cap V_1) \oplus (\tilde{F} \cap V_2) \subset \tilde{F}$, we have: $2n_1 = n_1 + n_2 \leq \dim \tilde{F}$. (**) Similarly $(\tilde{F} \cap W_1) \oplus (\tilde{F} \cap W_2) \subset \tilde{F}$ et $2m_1 = m_1 + m_2 \leq \dim \tilde{F}$. (***)

From (*), (**) and (***) follows that $2n_i = 2m_j = \dim \tilde{F}$ and that $(\tilde{F} \cap V_1) \oplus (\tilde{F} \cap V_2) = \tilde{F}$ and $(\tilde{F} \cap W_1) \oplus (\tilde{F} \cap W_2) = \tilde{F}$. \square

We can refine the two first points of the proposition as follows:

Proposition 9. *For every non negative integer n we have:*

- (1) $(\tilde{F}(n) \cap V_1) \oplus (\tilde{F}(n) \cap V_2) = \tilde{F}(n)$
- (2) $(\tilde{F}(n) \cap W_1) \oplus (\tilde{F}(n) \cap W_2) = \tilde{F}(n)$

Proof. We will just prove the first point, the proof of the second point being similar.

By induction on n : For $n = 0$ we have effectively: $\tilde{F}(0) = E = V_1 \oplus V_2$. Suppose the the result true for n . Evidently we have the inclusion: $(\tilde{F}(n+1) \cap V_1) \oplus (\tilde{F}(n+1) \cap V_2) \subset \tilde{F}(n+1)$. Let $a \in \tilde{F}(n+1)$. We can write $a = x + y$ with $x \in V_1$ and $y \in V_2$. Let us show then $x, y \in \tilde{F}(n+1)$.

As $a \in \tilde{F}(n+1) \subset \tilde{F}(n)$ and $\tilde{F}(n)$ is homogeneous with respect to $V_1 \oplus V_2$ we have: $x, y \in \tilde{F}(n)$.

By definition of $\tilde{F}(n+1)$, a we can write $a = x_{ij} + y_{ij}$ with $x_{ij} \in \tilde{F}(n) \cap V_i$ and $y_{ij} \in \tilde{F}(n) \cap W_j$. We deduce that x is an element of $\tilde{F}(n+1) = \bigcap_{i,j} ((\tilde{F}(n) \cap V_i) + (\tilde{F}(n) \cap W_j))$ by writing: $x = x + 0 = x + 0 = (x_{21} - y) + y_{21} = (x_{22} - y) + y_{22}$. A similar reflection shows that $y \in \tilde{F}(n+1)$. \square

We will see in the following that one can decompose canonically $F(n)$.

Let's write $e = id_{\{1,2\}}$ and $\tau = (12)$ the elements of the group \mathcal{S}_2 of the permutations of the set $\{1,2\}$. We will write for $i = 1, 2$, $\bar{i} := \tau(i)$. For $\sigma \in \mathcal{S}_2$, we write $\bar{\sigma}$ the element of \mathcal{S}_2 such that $\{\sigma, \bar{\sigma}\} = \mathcal{S}_2$.

Definition 3. Let $F_\sigma(0) = 0$ and $F_\sigma(n+1) = \sum_i ((F_\sigma(n) + V_i) \cap (F_\sigma(n) + W_{\sigma(i)}))$.

One can see that $(F_\sigma(n))_n$ is an increasing sequence of subvectorspaces of E , and so finally stationary (as E is finite-dimensional). Let's write $F_\sigma(\infty)$ or simply F_σ the space $\sum_n F_\sigma(n)$ i.e. the maximal element of this sequence.

Let's remark on the other hand that lemma 4 implies that $F_e(1) = (V_1 \cap W_1) \oplus (V_2 \cap W_2)$, $F_\tau(1) = (V_1 \cap W_2) \oplus (V_1 \cap W_2)$ and $F(1) = F_e(1) \oplus F_\tau(1)$.

Proposition 10. $\forall n, \theta(F_\sigma(n+1)) \subset F_\sigma(n)$.

Proof. By induction: It is true for $n = 0$. Suppose its true up to order n . Let $x \in F_\sigma(n+1)$, $y \in V_1$, $z \in F_\sigma(n+1)$, $t \in V_2$, $x' \in F_\sigma(n+1)$, $y' \in W_1$, $z' \in F_\sigma(n+1)$, $t' \in W_2$, such that $x + y = x' + y'$ et $z + t = z' + t'$.

Let us show that $\theta(x + y + z + t) \in F_\sigma(n+1)$. Let us recall first that $\theta(V_i) \subset V_{\tau(i)}$ and $\theta(W_j) \subset W_{\tau(j)}$. We have consequently: $\theta(x) + \theta(y) = \theta(x') + \theta(y') \in (F_\sigma(n) + V_2) \cap (F_\sigma(n) + W_{\sigma(2)})$ and $\theta(z) + \theta(t) = \theta(z') + \theta(t') \in (F_\sigma(n) + V_1) \cap (F_\sigma(n) + W_{\sigma(1)})$. This gives us $\theta(x + y + z + t) = \theta(x) + \theta(y) + \theta(z) + \theta(t) \in F_\sigma(n+1)$. \square

We will need the following lemma:

Lemma 11. Let A_0, A, B_0, B be four vector subspaces of E such that $A_0 \subset A$ et $B_0 \subset B$. We have then

$$(A + B_0) \cap (A_0 + B) = A_0 + B_0 + (A \cap B).$$

Proof. The inclusion " \supset " is clear, as every $A + B_0$, $A_0 + B$ contains every A_0 , B_0 , $A \cap B$.

For the inclusion " \subset " let $x \in A$, $y_0 \in B_0$, $x_0 \in A_0$, $y \in B$ such that $x + y_0 = x_0 + y$. One deduces $x - x_0 = y - y_0 \in A \cap B$. So $x + y_0 = x_0 + y_0 + (x - x_0) \in A_0 + B_0 + (A \cap B)$. \square

Proposition 12. (1) $F_\sigma(n)$ is co-homogeneous with respect to the direct sum $V_1 \oplus V_2$ or equivalently $(F_\sigma(n) + V_1) \cap (F_\sigma(n) + V_2) = F_\sigma(n)$.
 (2) $F_\sigma(n)$ is co-homogeneous with respect to the direct sum $W_1 \oplus W_2$ or equivalently $(F_\sigma(n) + W_1) \cap (F_\sigma(n) + W_2) = F_\sigma(n)$.

Proof. We will prove the first point, the proof for the second being similar.

By induction: For $n = 0$ its clear. Suppose the result true at the order n . It is evident that $F_\sigma(n) \subset (F_\sigma(n) + V_1) \cap (F_\sigma(n) + V_2)$.

Let's prove the other inclusion: We have:

$$\begin{aligned} F_\sigma(n+1) + V_1 &= \sum_i ((F_\sigma(n) + V_i) \cap (F_\sigma(n) + W_{\sigma(i)})) + V_1 \\ &\subset \underbrace{F_\sigma(n) + V_1}_A + \underbrace{(F_\sigma(n) + V_2) \cap (F_\sigma(n) + W_{\sigma(2)})}_{B_0}. \end{aligned}$$

Similarly

$$F_\sigma(n+1) + V_2 \subset \underbrace{(F_\sigma(n) + V_1) \cap (F_\sigma(n) + W_{\sigma(1)})}_{A_0} + \underbrace{F_\sigma(n) + V_2}_B.$$

By application of lemma 11 and the induction hypothesis we obtain:

$$\begin{aligned} (F_\sigma(n+1) + V_1) \cap (F_\sigma(n+1) + V_2) &\subset F_\sigma(n+1) + (F_\sigma(n) + V_1) \cap (F_\sigma(n) + V_2) \\ &\stackrel{\text{ind. hyp.}}{\subset} F_\sigma(n+1) + F_\sigma(n) \\ &\subset F_\sigma(n+1). \end{aligned}$$

\square

Proposition 13. $\forall n, F_\sigma(n) \cap F_{\bar{\sigma}}(1) = \{0\}$.

Proof. Let's make the proof for $\sigma = e$, the case $\sigma = \tau$ being analogous.

By induction: It is true up to order $n = 0$. Suppose it is true up to order n : Let $x \in F_e(n), y \in V_1, z \in F_e(n), t \in V_2, x' \in F_e(n), y' \in W_1, z' \in F_e(n), t' \in W_2, \gamma \in V_1 \cap W_2, \delta \in V_2 \cap W_1$ such that $x + y = x' + y', z + t = z' + t'$ and $(x + y) + (z + t) = \gamma + \delta \in F_e(n + 1) \cap F_\tau(1)$.

One has then $x + (y - \gamma) \in F_e(n) + V_1, -(z + (t - \delta)) \in F_e(n) + V_2$, et $x + (y - \gamma) = -(z + (t - \delta))$. By application of proposition 12 one obtains $y - \gamma \in F_e(n)$ and $t - \delta \in F_e(n)$. One deduces: $x + y = (x + (y - \gamma)) + \gamma \in F_e(n) + (V_1 \cap W_2)$ and also $z + t = (z + (t - \delta)) + \delta \in F_e(n) + (V_2 \cap W_1)$. Analogously one proves that $x' + y' = (x' + (y' - \delta)) + \delta \in F_\sigma(n) + (V_2 \cap W_1)$ and $z' + t' = (z' + (t' - \gamma)) + \gamma \in F_\sigma(n) + (V_1 \cap W_2)$. By a new application of proposition 12 (possible by the fact that $V_1 \cap W_2 \subset V_1$ and $V_2 \cap W_1 \subset V_2$) one obtains that $x + y = x' + y' \in F_e(n)$ and similarly $z + t = z' + t' \in F_e(n)$. So $(x + y) + (z + t) \in F_e(n) \cap F_\tau(1)$. By induction hypothesis one has so $(x + y) + (z + t) = 0$. \square

Corollary 14. If $n \geq 1$ then $F_\sigma(n) \cap F(1) = F_\sigma(1)$

Proof. It is clear that $F_\sigma(1) \subset F_\sigma(n) \cap F(1)$. For the other inclusion, let's remark first: $F_\sigma(n) \cap F(1) = F_\sigma(n) \cap (F_\sigma(1) \oplus F_{\bar{\sigma}}(1))$. Let $x = a + b \in F_\sigma(n)$ with $a \in F_\sigma(1)$ and $b \in F_{\bar{\sigma}}(1)$. $x - a = b \in F_\sigma(n) \cap F_{\bar{\sigma}}(1) = \{0\}$. So we have $x \in F_\sigma(1)$. \square

Proposition 15. $F_e(n) \cap F_\tau(n) = \{0\}$

Proof. By induction: It is true for $n = 0$. Suppose it's true up to order n . Let $x \in F_e(n + 1) \cap F_\tau(n + 1)$, one deduces then $\theta(x) \in \theta(F_e(n + 1)) \cap \theta(F_\tau(n + 1)) \stackrel{\text{prop. 10}}{\subset} F_e(n) \cap F_\tau(n) = \{0\}$. From this we obtain $x \in F(1) \cap F_e(n) \cap F_\tau(n) = (F(1) \cap F_e(n)) \cap (F(1) \cap F_\tau(n)) \stackrel{\text{corr. 14}}{=} F_e(1) \cap F_\tau(1) = \{0\}$. \square

Proposition 16. $\forall n, F_e(n) \oplus F_\tau(n) = F(n)$

Let's start by proving two lemma:

Lemma 17. $\forall i, n, F_\sigma(n) \subset V_i + W_{\bar{\sigma}(i)}$

Proof. Let's give the proof for $\sigma = \tau$. The proof is essentially the same in the case $\sigma = e$.

By induction on n : For $n = 0$ we have $F_\tau(0) = \{0\} \subset V_i + W_i$. Suppose the result true up to order n . $(F_\tau(n) + V_i) \cap (F_\tau(n) + W_i) \subset F_\tau(n) + V_i$ and $(F_\tau(n) + V_i) \cap (F_\tau(n) + W_i) \subset F_\tau(n) + W_i$. By summation of the two inclusions one obtains $F_\tau(n + 1) \subset F_\tau(n) + V_i + F_\tau(n) + W_i$. The latter is included in $V_i + W_i$ by induction hypothesis. \square

Lemma 18. $\forall n, i, j, V_i + W_j$ is homogeneous with respect to the (direct) sum $F_e(n) + F_\tau(n)$.

Proof. Let's make the proof for $i = j = 1$, the proof being similar in the other cases.

The inclusion $(V_1 + W_1) \cap F_e(n) + (V_1 + W_1) \cap F_\tau(n) \subset (V_1 + W_1) \cap (F_e(n) + F_\tau(n))$ being trivial, let us show the other inclusion: Let $\alpha \in F_e(n), \beta \in F_\tau(n)$ with $\alpha + \beta \in V_1 + W_1$. By the inclusion $F_\tau(n) \subset V_1 + W_1$ obtained

by the preceding lemma one has: $\beta \in F_\tau(n) \cap (V_1 + W_1)$. As $\alpha + \beta \in V_1 + W_1$ and $\beta \in V_1 + W_1$ one has $\alpha = (\alpha + \beta) - \beta \in F_e(n) \cap (V_1 + W_1)$. \square

Proof. proposition 16: By induction on n . For $n = 0$ it is evident. Suppose the result proved up to order n .

Let us recall that $F(n+1) = \ker \theta^{n+1}$. Let $x \in F(n+1)$. By induction hypothesis there exists $\alpha \in F_e(n)$, $\beta \in F_\tau(n)$ such that $\theta(x) = \alpha + \beta$. Set $v_{ij} = (p_i \circ q_j)(x)$ and $w_{ij} = (q_j \circ p_i)(x)$. Let us remark that $v_{ij} \in V_i$ and $w_{ij} \in W_j$. Recall that $w_{ij} = \theta(x) + v_{ij} = \alpha + \beta + v_{ij}$. So one has more precisely $w_{ij} \in (F_e(n) + F_\tau(n) + V_i) \cap W_j$. As in the proof of proposition 5 let us remark that $x = \sum_{i,j} w_{ij}$. If one proves that $w_{ij} \in F_e(n+1) + F_\tau(n+1)$ the proposition is proved.

As $\alpha + \beta = -v_{ij} + w_{ij} \in (F_e(n) \oplus F_\tau(n)) \cap (V_i + W_j)$ one can apply lemma 18 in order to obtain that $\alpha_{ij} \in V_i, \alpha'_{ij} \in W_j$ such that $\alpha = \alpha_{ij} + \alpha'_{ij}$ and $\beta_{ij} \in V_i, \beta'_{ij} \in W_j$ such that $\beta = \beta_{ij} + \beta'_{ij}$. One has: $\alpha'_{ij} = \alpha - \alpha_{ij} \in (F_e(n) + V_i) \cap W_j \subset F_e(n+1)$ and $\beta'_{ij} = \beta - \beta_{ij} \in (F_\tau(n) + V_i) \cap W_j \subset F_\tau(n+1)$. On the other hand $W_j \ni w_{ij} - \alpha'_{ij} - \beta'_{ij} = \alpha_{ij} + (v_{ij} + \beta_{ij}) \in F_e(n) + V_i$ and so $w_{ij} - \alpha'_{ij} - \beta'_{ij} \in (F_e(n) + V_i) \cap W_j \subset F_e(n+1)$. Finally $w_{ij} = (w_{ij} - \alpha'_{ij} - \beta'_{ij}) + \alpha'_{ij} + \beta'_{ij} \in F_e(n+1) + F_\tau(n+1)$, and so $x = \sum_{i,j} w_{ij} \in F_e(n+1) + F_\tau(n+1)$. \square

Proposition 19. (1) $F_\sigma(n)$ is homogeneous with respect to the sum $V_1 \oplus V_2$, equivalently $F_\sigma(n) = (F_\sigma(n) \cap V_1) \oplus (F_\sigma(n) \cap V_2)$.
 (2) $F_\sigma(n)$ is homogeneous with respect to the sum $W_1 \oplus W_2$, equivalently $F_\sigma(n) = (F_\sigma(n) \cap W_1) \oplus (F_\sigma(n) \cap W_2)$

Proof. Let us prove the first point. The proof of the second is similar. Let $x \in F_\sigma(n)$, $x = y + z$ with $y \in V_1$ and $z \in V_2$. We have then $y = x - z \in V_1 \cap (F_\sigma(n) + V_2) \subset (F_\sigma(n) + V_1) \cap (F_\sigma(n) + V_2) \stackrel{\text{Prop. 12}}{=} F_\sigma(n)$. From this $y \in F_\sigma(n) \cap V_1$. In the same way $z \in F_\sigma(n) \cap V_2$. As a conclusion $F_\sigma(n) = (F_\sigma(n) \cap V_1) \oplus (F_\sigma(n) \cap V_2)$. \square

Proposition 20. $\forall n, (F_\sigma(n) \cap V_i) \oplus (F_\sigma(n) \cap W_{\bar{\sigma}(i)}) = F_\sigma(n)$

Proof. We have from proposition 13 $(F_\sigma(n) \cap V_i) \cap (F_\sigma(n) \cap W_{\bar{\sigma}(i)}) = \{0\}$. Let us write $n_i := \dim F_\sigma(n) \cap V_i$ and $m_j := \dim F_\sigma(n) \cap W_j$. We have from the preceding remark that $n_i + m_{\bar{\sigma}(i)} \leq \dim F_\sigma(n)$ (*). From proposition 19 one has $n_1 + n_2 = \dim F_\sigma(n)$ and $m_1 + m_2 = \dim F_\sigma(n)$. By summing the two equalities it is necessary that (*) is an equality and so $(F_\sigma(n) \cap V_i) \oplus (F_\sigma(n) \cap W_{\bar{\sigma}(i)}) = F_\sigma(n)$. \square

Proposition 21. If A, B vector subspaces of E are homogeneous with respect to the sum $\oplus_{i \in I} F_i = E$ then $A + B$ and $A \cap B$ are homogeneous with respect to the sum $\oplus_{i \in I} F_i$.

Proof. " $A + B$ ": Equivalently one has: $\bigoplus_i (F_i \cap (A + B)) \subset A + B$. Let us show the other inclusion. Let $x \in A + B = (\bigoplus_i (F_i \cap A)) + (\bigoplus_i (F_i \cap B))$. So one has $x = \sum_i x_i + \sum_i x'_i$ with $x_i \in F_i \cap A$ and $x'_i \in F_i \cap B$. By writing $x = \sum_i (x_i + x'_i)$ one sees that $x \in \bigoplus_i (F_i \cap (A + B))$.

" $A \cap B$ ": Evidently one has: $\bigoplus_i (F_i \cap (A \cap B)) \subset A \cap B$. For the other inclusion let $x \in A \cap B = (\bigoplus_i (F_i \cap A)) \cap (\bigoplus_i (F_i \cap B))$, $x = \sum_i x_i = \sum_i x'_i$

with $x_i \in F_i \cap A$ and $x'_i \in F_i \cap B$. By unicity of the decomposition of x with respect to the direct sum $\bigoplus_i F_i$ it is clear that $\forall i, x_i = x'_i$ and so that $x \in \bigoplus_i (F_i \cap (A \cap B))$. \square

Proposition 22. *For every element V of the lattice generated by V_1, V_2, W_1 and W_2 one has: $V = (V \cap F_e) \oplus (V \cap F_\tau) \oplus (V \cap \tilde{F})$*

Proof. Due to proposition 21 it is enough to prove that V_1, V_2, W_1 and W_2 are homogeneous with respect to the sum: $E = F_e \oplus F_\tau \oplus \tilde{F}$.

Let us prove for this purpose the lemma:

Lemma 23. *Let E, E_j et F_i be vector spaces. If $E = E_1 \oplus E_2$ with $\forall i, F_i = (F_i \cap E_1) \oplus (F_i \cap E_2)$, then $E_j \cap \bigoplus_i F_i = \bigoplus_i (E_j \cap F_i)$ for $j = 1, 2$.*

Proof. $\bigoplus_i F_i = \bigoplus_i (F_i \cap (E_1 \oplus E_2)) = \bigoplus_i ((F_i \cap E_1) \oplus (F_i \cap E_2)) = \bigoplus_i (F_i \cap E_1) \oplus \bigoplus_i (F_i \cap E_2)$. But $\bigoplus_i (F_i \cap E_j) \subset (\bigoplus_i F_i) \cap E_j$. As $((\bigoplus_i F_i) \cap E_1) \oplus ((\bigoplus_i F_i) \cap E_2) \subset \bigoplus_i F_i$, the inclusions in this proof are necessarily equalities. So $\bigoplus_i (F_i \cap E_j) = (\bigoplus_i F_i) \cap E_j$. \square

end of proof of proposition 22: By applying the lemma for $\forall i, E_i = V_i$ (respectively $\forall i, E_i = W_i$) proposition 19 and proposition 8 show that V_1, V_2, W_1 and W_2 are homogeneous with respect to the sum decomposition: $E = F_e \oplus F_\tau \oplus \tilde{F}$. \square

2.3. Reflexive case. Suppose that E, V_1, V_2 are finite-dimensional vector-spaces such that $E = V_1 \oplus V_2$ and suppose that E carries a non degenerate reflexive form a . We have seen that $(E, V_1, V_2, V_1^\perp, V_2^\perp)$ is a decomposition of E into two direct sums. Suppose $F(n), F, F_\sigma(n), F_\sigma, \tilde{F}(n)$ and \tilde{F} defiend as before.

Let's prove the following proposition:

Proposition 24.

$$F = F_e \oplus^\perp F_\tau \oplus^\perp \tilde{F}$$

Proof. For $\sigma \in \mathcal{S}_2$ let $\tilde{F}_\sigma(0) := E$ and $\tilde{F}_\sigma(n+1) := \bigcap_i ((\tilde{F}_\sigma(n) \cap V_i) + (\tilde{F}_\sigma(n) \cap W_{\sigma(i)}))$. The sequence $\tilde{F}_\sigma(n)$ is decreasing and so stationary in finite dimensions. Note $\tilde{F}_\sigma := \bigcap_n \tilde{F}_\sigma(n)$.

By induction it is easy to see that² $\forall n, \forall \sigma \in \mathcal{S}_2, F_\sigma(n)^\perp = \tilde{F}_\sigma(n)$.

By writing the definition of $\tilde{F}(n)$ and $\tilde{F}_\sigma(n)$ it is easy to see by induction that $\forall n, \forall \sigma, \tilde{F}(n) \subset \tilde{F}_\sigma(n)$, from which we obtain $\forall \sigma, \tilde{F} \subset \tilde{F}_\sigma$.

In order to finish the proof lets show the following lemma:

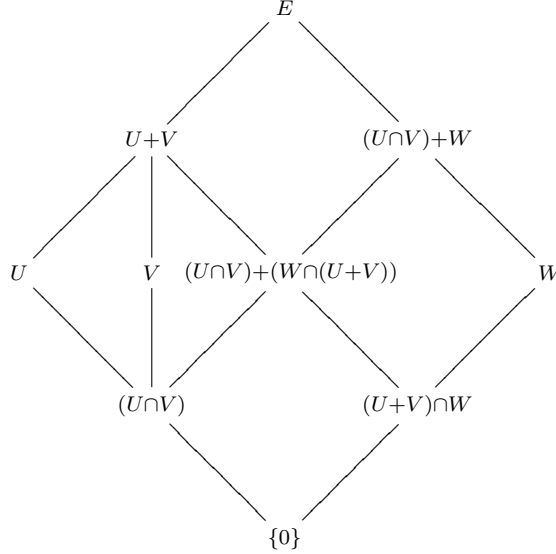
Lemma 25. *For $\sigma \in \mathcal{S}_2$ we have: $\forall n, F_{\bar{\sigma}} \subset \tilde{F}_\sigma(n)$*

Proof. By induction on n : It is clear for $n = 0$. For $n+1$ we have: $\tilde{F}_\sigma(n+1) = \bigcap_i ((\tilde{F}_\sigma(n) \cap V_i) + (\tilde{F}_\sigma(n) \cap W_{\sigma(i)})) \subset \bigcap_i ((F_{\bar{\sigma}} \cap V_i) + (F_{\bar{\sigma}} \cap W_{\sigma(i)}))$ by induction hypothesis. The latter expression is equal to $F_{\bar{\sigma}}$ by proposition 20. \square

end of the proof of proposition 24: As $\dim(F_\sigma) + \dim(\tilde{F}_\sigma) = \dim(E) = \dim(F_\sigma) + \dim(F_{\bar{\sigma}}) + \dim(\tilde{F})$ (because F_σ and \tilde{F}_σ are orthogonal, respectively by proposition 22) we have $\dim(\tilde{F}_\sigma) = \dim(F_{\bar{\sigma}}) + \dim(\tilde{F})$. By the inclusion $F_{\bar{\sigma}} \oplus \tilde{F} \subset \tilde{F}_\sigma$ we must have $F_{\bar{\sigma}} \oplus \tilde{F} = \tilde{F}_\sigma$. \square

²By using again the fact that $(A+B)^\perp = A^\perp \cap B^\perp$ and $(A \cap B)^\perp = A^\perp + B^\perp$ for A and B vector subspaces of E

2.4. Sublattice "with 5 direct sums". It is known that the lattice generated by the three vector subspaces of E : U, V, W such that $E = U \oplus W = V \oplus W$ has the following structure:



The construction applies to the lattice T generated by the 4 subspaces of E , V_1, V_2, W_1, W_2 such that $E = V_1 \oplus V_2 = W_1 \oplus W_2 = V_1 \oplus W_2 = W_1 \oplus V_2$, in the following way:

We can choose for (U, V, W) the triple (V_1, W_1, V_2) or (V_1, W_1, W_2) . Note that then in the first case: $T_1 := (V_1 \cap W_1) + (V_2 \cap (V_1 + W_1))$ and in the second: $U_1 := (V_1 \cap W_1) + (W_2 \cap (V_1 + W_1))$.

The interval $[V_1 \cap W_1, V_1 + W_1]$ is a sub-lattice T' of T which contains in particular the elements $V'_1 := V_1/(V_1 \cap W_1)$, $W'_1 := W_1/(V_1 \cap W_1)$, $T'_1 := T_1/(V_1 \cap W_1)$ et $U'_1 := U_1/(V_1 \cap W_1)$ verifying:

$$V'_1 \oplus W'_1 = V'_1 \oplus T'_1 = V'_1 \oplus U'_1 = W'_1 \oplus T'_1 = W'_1 \oplus U'_1$$

On the other hand it is possible that $T'_1 \cap U'_1 \neq \{0\}$ (as well as $T'_1 + U'_1 \neq (V_1 + W_1)/(V_1 \cap W_1)$).

Note is particular that T' contains two sublattices of type M_3 : The one constructed on the elements $\{\{0\}, E, V'_1, W'_1, T'_1\}$ and the one given by the elements $\{\{0\}, E, V'_1, W'_1, U'_1\}$.

The data of T'_1 is equivalent to the data of an isomorphism i of V'_1 onto W'_1 , and the data of U'_1 of a second isomorphism j of V'_1 onto W'_1 . the conjugation class in $Gl(V'_1)$ of $j^{-1} \circ i$ is then an invariant of the lattice. We can compare this result to the operators that Gelfand and Ponomarev used in their paper [GP].

2.5. Example. In this paragraph we are going to study the structure of the lattice generated by four finite-dimensional vector spaces V_1, V_2, W_1, W_2 such that $E = V_1 \oplus V_2 = W_1 \oplus W_2 = V_1 \oplus W_2 = W_1 \oplus V_2$ supposing that $\theta_{\mathcal{V}}^2 = 0$ for $\mathcal{V} = (E, V_1, V_2, W_1, W_2)$.

Lemma 26. *On a: $(V_1 + W_1) \cap V_2 = (V_1 + W_1) \cap W_2 \subset V_2 \cap W_2$ et $(V_2 + W_2) \cap V_1 = (V_2 + W_2) \cap W_1 \subset V_1 \cap W_1$*

Proof. It is clear that $(V_1 + W_1) \cap V_2 \subset (V_1 + W_1) \cap (V_2 + W_2) = \text{im}\theta \subset \ker \theta = (V_1 \cap W_1) \oplus (V_2 \cap W_2)$. From which one can see that $(V_1 + W_1) \cap V_2 \subset \ker \theta \cap V_2 = V_2 \cap W_2$. So $(V_1 + W_1) \cap V_2 \subset (V_1 + W_1) \cap W_2$ and similarly $(V_1 + W_1) \cap W_2 \subset (V_1 + W_1) \cap V_2$, which proves the first assertion. The proof of the second one is similar. \square

Note $X_0 = \{0\}$, $X_1 = (V_2 + W_2) \cap V_1$, $X_2 = V_1 \cap W_1$, $X_3 = V_1$ et $Y_0 = \{0\}$, $Y_1 = (V_1 + W_1) \cap V_2$, $Y_2 = V_2 \cap W_2$, $Y_3 = V_2$.

As $X_0 \subset X_1 \subset X_2 \subset X_3 = V_1$ and $Y_0 \subset Y_1 \subset Y_2 \subset Y_3 = V_2$ and $V_1 \cap V_2 = \{0\}$, it is easy to see that the lattice \mathcal{T}_0 generated by the X_i and the Y_j for $i, j = 0, 1, 2, 3$ is precisely the set $\{X_i \oplus Y_j \mid i, j = 0, 1, 2, 3\}$, ordered by inclusion.

It is easy to verify that $X_i \oplus Y_j = (X_i \oplus V_2) \cap (V_1 \oplus Y_j)$ and so the lattice \mathcal{T}_0 can be written as well:

Note $X'_i = X_i \oplus V_2$ et $Y'_j = V_1 \oplus Y_j$. We have then: $X'_0 = V_2$, $X'_1 = ((V_2 + W_2) \cap V_1) + V_2$, $X'_2 = (V_1 \cap W_1) + V_2$, and $X'_3 = V_1 + V_2$.

Let's verify: $X'_1 = V_2 + W_2$. It is clear that $X'_2 \subset V_2 + W_2$ inversely if $x \in V_2$ and $y \in W_2$, $x+y$ can be written uniquely $a+b$ with $a \in V_1$ and $b \in V_2$, so $a = ((x-b)+y) + b \in (V_2 + W_2) \cap V_1$ and so $a+b \in ((V_2 + W_2) \cap V_1) + V_2$. So $X'_1 = V_2 + W_2$.

We have as well: $Y'_0 = V_1$, $Y'_1 = V_1 + W_1$, $Y'_2 = (V_2 \cap W_2) + V_1$, et $Y'_3 = V_1 + V_2$.

The underlying set of the lattice \mathcal{T}_0 is so: $\{X'_i \cap Y'_j \mid i, j = 0, 1, 2, 3\}$.

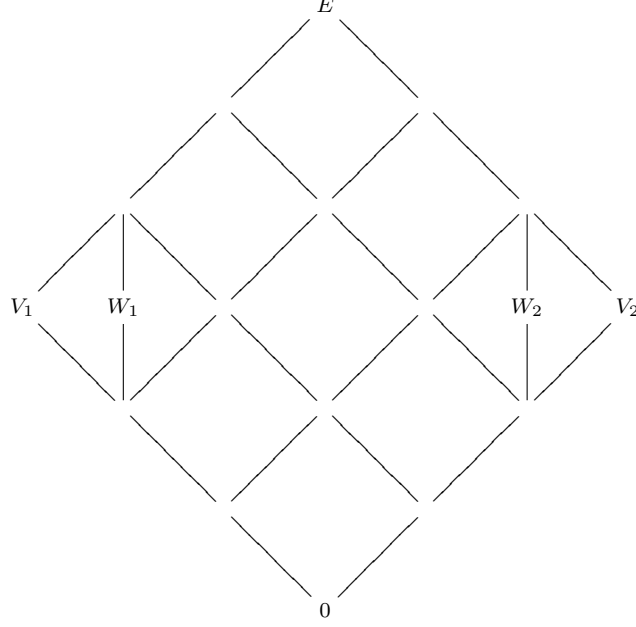
We are going to prove that $\mathcal{T} = \mathcal{T}_0 \cup \{W_1, W_2\}$ is a lattice. Let us verify that \mathcal{T} is stable by intersection and sum.

Verify that $(X_i \oplus Y_j) + W_1 \in \mathcal{T}$: If $j = 0$ and $i = 0, 1, 2$ it is clear that $(X_i \oplus Y_j) + W_1 = W_1 \in \mathcal{T}$. If $j = 0$ and $i = 3$, $(X_i \oplus Y_j) + W_1 = V_1 + W_1 \in \mathcal{T}$. If $j \geq 1$, $(X_i \oplus Y_j) + W_1 = (Y_1 + W_1 + X_i + Y_j)$. By a similar argument to the one which allowed us to have before: $((V_2 + W_2) \cap V_1) + V_2 = V_2 + W_2$, one can prove $Y_1 + W_1 = ((V_1 + W_1) \cap V_2) + W_1 = V_1 + W_1 \in \mathcal{T}_0$, and so $Y_1 + W_1 + X_i + Y_j \in \mathcal{T}_0 \subset \mathcal{T}$.

By using the second representation of \mathcal{T}_0 we can show for every i and j , $(X'_i \cap Y'_j) \cap W_1 \in \mathcal{T}$. The only delicate point is to verify that $((V_1 \cap W_1) + V_2) \cap W_1 = V_1 \cap W_1$. Let's do it: It is clear that $V_1 \cap W_1 \subset ((V_1 \cap W_1) + V_2) \cap W_1$, inversely let $x \in V_1 \cap W_1$, $y \in V_2$ and $z \in W_1$ such that $x + y = z$. We have then: $y = z - x \in V_2 \cap W_1 = \{0\}$, and so $z = x \in V_1 \cap W_1$.

In conclusion we can state:

Theorem 27. *The structure of the lattice generated by the four finite-dimensional vector spaces V_1, V_2, W_1, W_2 such that $E = V_1 \oplus V_2 = W_1 \oplus W_2 = V_1 \oplus W_2 = W_1 \oplus V_2$ and supposing that $\theta_{\mathcal{V}}^2 = 0$ for $\mathcal{V} = (E, V_1, V_2, W_1, W_2)$ is given by the following diagram:*



3. APPLICATION TO REPRESENTATION THEORY

3.1. Preliminaries.

3.1.1. *General case.* We will note $\mathfrak{gl}(V_1, V_2, W_1, W_2)$ the set of $a \in \mathfrak{gl}(E)$ such that $aV_i \subset V_i$ et $aW_j \subset W_j$. It is easy to see that $\mathfrak{gl}(V_1, V_2, W_1, W_2)$ is a sub Lie-algebra of $\mathfrak{gl}(E)$. Let \mathfrak{g} be a sub Lie-algebra of $\mathfrak{gl}(V_1, V_2, W_1, W_2)$. We have for all A, B vector subspaces of E such that $\mathfrak{g}A \subset A$ et $\mathfrak{g}B \subset B$: $\mathfrak{g}(A+B) \subset (A+B)$ et $\mathfrak{g}(A \cap B) \subset (A \cap B)$. So we have, as \mathfrak{g} leaves invariant V_1, V_2, W_1 et W_2 , \mathfrak{g} leaves invariant every element of the lattice generated from V_1, V_2, W_1 et W_2 by intersection and sum.

It is easy to see that the projections $p_{V_i}^{V_j}$ and $p_{W_i}^{W_j}$ commute to the action of \mathfrak{g} : $\forall a \in \mathfrak{g}, ap_{V_i}^{V_j} = p_{V_i}^{V_j}a$ and $ap_{W_i}^{W_j} = p_{W_i}^{W_j}a$. So every element of the associative unitary algebra A generated by the $p_{V_i}^{V_j}$ and the $p_{W_i}^{W_j}$ commutes to every $a \in \mathfrak{g}$. As an example $\theta = [p_{W_1}^{W_2}, p_{V_1}^{V_2}]$ commutes to every $a \in \mathfrak{g}$.

Lemma 28. *The data of two supplementary vector-spaces V_1 et V_2 stable for the action of a linear Lie algebra \mathfrak{g} is equivalent to the data of an endomorphism L commuting with the action of \mathfrak{g} , verifying $L^2 = I$. V_1 et V_2 are then the proper subspaces of L associated to the eigenvalues 1 and -1 .*

Proof. In fact it is easy to see that the endomorphism $L = p_{V_1}^{V_2} - p_{V_2}^{V_1}$ is of square identity and commutes to the action of \mathfrak{g} . Inversely if an endomorphism L is such that $L^2 = I$ and commutes to the action of \mathfrak{g} , it admits the proper values 1 and/or -1 . The corresponding eigenspaces are supplementary and stable for the action of \mathfrak{g} . \square

3.1.2. *Reflexive case.* We recall that in the reflexive case we suppose that there exists a non degenerate reflexive form $\langle \cdot, \cdot \rangle$ such that $\forall a \in \mathfrak{g}, \forall x, y \in E$, we have: $\langle ax, y \rangle + \langle x, ay \rangle = 0$.

Recall as well that if V is a subspace of E which is \mathfrak{g} -invariant then V^\perp is invariant as well. We suppose here that $W_1 = V_1^\perp$, $W_2 = V_2^\perp$. These two spaces are supplementary and invariant.

Lemma 29. *Let L^* be the adjoint with respect to a reflexive form of the endomorphism $L = p_{V_1}^{V_2} - p_{V_2}^{V_1}$ which commutes to the action of \mathfrak{g} and is such that $L^2 = I$. Then L^* is of square identity, commutes to the action of \mathfrak{g} and one has:*

$$L^* = p_{V_2^\perp}^{V_1^\perp} - p_{V_1^\perp}^{V_2^\perp}$$

Proof. Let's note $L' := p_{V_2^\perp}^{V_1^\perp} - p_{V_1^\perp}^{V_2^\perp}$ and let us show that $\forall v, w \in E$, $\langle Lv, w \rangle = \langle v, L'w \rangle$,

We write for $x \in V_1, x' \in V_2, y \in V_1^\perp, y' \in V_2^\perp$,

$$\begin{aligned} \langle L(x + x'), y + y' \rangle &= \langle x - x', y + y' \rangle \\ &= \langle x, y' \rangle - \langle x', y \rangle \\ &= \langle x + x', -y + y' \rangle \\ &= \langle x + x', L'(y + y') \rangle \end{aligned}$$

As a consequence $L^* = L'$. \square

Let's remark that $L = -L^*$ for $L = p_{V_1}^{V_2} - p_{V_2}^{V_1}$ is equivalent to have $V_1 = V_1^\perp$ and $V_2 = V_2^\perp$. It is the same to impose $\langle Lx, Ly \rangle = -\langle x, y \rangle$ for $x, y \in E$ i.e. L is anti-hermitian with respect to the reflexive form.

The data of $L \in \text{End}(E)$ such that $L^2 = Id$ and of a reflexive form for which L is anti-hermitian is also called a para-Kähler structure.

We recall that the reflexive representation $\mathfrak{g} \subset \mathfrak{gl}(E)$ is called *weakly irreducible* if any invariant subspace $V \subset E$ is either $\{0\}$, E , or is degenerate i.e. $V \cap V^\perp \neq \{0\}$.

As we saw in paragraph 2.3, if $W_1 = V_1^\perp$ and $W_2 = V_2^\perp$, we have in the weakly irreducible case and if V_1 et V_2 are different from $\{0\}$ necessarily $E = F_e$. In fact if two of the three spaces F_e, F_τ, \tilde{F} are non trivial then E is not weakly irreducible. The more in the case $E = F_\tau$ or $E = \tilde{F}$, the fact that $E = V_1 \oplus V_1^\perp$ would imply that if E is non trivial, E is not weakly irreducible.

Proposition 30. *In the case the representation $E = V_1 \oplus V_2$ is weakly irreducible and if V_1 and V_2 are different from $\{0\}$, V_2 identifies (as a representation) to the dual V_1^* of V_1 .*

Proof. It identifies by the map

$$\begin{aligned} V_2 &\rightarrow V_1^* \\ v' &\mapsto (w \mapsto \langle v', w \rangle) \end{aligned}$$

which is injective by the fact that $V_1 \cap V_2^\perp = \{0\}$ and surjective for dimension reasons. In fact we have $V_1 \oplus V_2^\perp = V_1 \oplus V_2$ implies that $\dim(V_2) = \dim(V_2^\perp)$. From this we obtain $\dim(V_2) = \frac{1}{2}\dim(E)$ and similarly $\dim(V_1) = \dim(E) - \frac{1}{2}\dim(E) = \frac{1}{2}\dim(E)$. As $\dim(V_1^*) = \dim(V_1)$, we have: $\dim(V_1^*) = \dim(V_2)$. \square

3.2. Main result. The following result could be formulated thanks to a suggestion of Martin Olbrich. He communicated to us a direct proof of the result 32, which we had established for pseudo-riemannian holonomy algebras only.

Theorem 31. *If E is a representation admitting two decompositions into supplementary sub-representations $E = V_1 \oplus V_2 = W_1 \oplus W_2$, then, noting $E_{(L,\lambda)}$ the generalized eigenspace associated to the eigenvalue λ for the operator L , we have:*

- (i) $F_e = E_{(L,-1)} \oplus E_{(L,1)}$ as a representation for the invariant operator $L = p_{V_1}^{V_2} - p_{W_2}^{W_1}$. The more we have $V_1 \cap W_1 \subset E_{(L,1)}$ and $V_2 \cap W_2 \subset E_{(L,-1)}$
- (ii) $F_\tau = E_{(L',-1)} \oplus E_{(L',1)}$ as a representation for the invariant operator $L' = p_{V_1}^{V_2} - p_{W_1}^{W_2}$. The more we have $V_1 \cap W_2 \subset E_{(L',1)}$ et $V_2 \cap W_1 \subset E_{(L',-1)}$

When E is in addition reflexive and $W_j = V_j^\perp$, then

- (i) L is anti-self-adjoint with respect to the reflexive form, $E_{(L,-1)}$ and $E_{(L,1)}$ are totally isotropic and their direct sum is non degenerate.
- (ii) L' is self-adjoint with respect to the reflexive form, $E_{(L',-1)}$ and $E_{(L',1)}$ are non degenerate and orthogonal.

Proof. It follows from the fact that the spaces F_e , F_τ and \tilde{F} are homogeneous, that $L = p_{V_1}^{V_2} - p_{W_2}^{W_1}$ (and similarly $L' = p_{V_1}^{V_2} - p_{W_1}^{W_2}$) is an endomorphism of each of these spaces.

For $\sigma = e$ or τ note $P_\sigma(X) = \Pi_{\lambda \in \Lambda_\sigma} P_{\sigma,\lambda}^{n_\lambda}(X)$ the minimal polynomial of L restricted to F_σ and similarly $\tilde{P}(X) = \Pi_{\lambda \in \tilde{\Lambda}} \tilde{P}_\lambda^{n_\lambda}(X)$ the minimal polynomial of L restricted to \tilde{F} .

F_σ decomposes into the generalized eigenspaces $F_{\sigma(L,\lambda)} := \ker(P_{\sigma,\lambda}^{n_\lambda}(L)|_{F_\sigma})$. and \tilde{F} decomposes into the generalized eigenspaces $\tilde{F}_{(L,\lambda)} := \ker(\tilde{P}_\lambda^{n_\lambda}(L)|_{\tilde{F}})$.

Let's make the convention that $P_{\sigma,\lambda}(X) = X + \lambda$ and $\tilde{P}_\lambda(X) = X + \lambda$ for $\lambda = 0, -1, 1$.

It is immediate that: $V_1 \cap W_1 \subset F_{e,(L,1)}$ and $V_2 \cap W_2 \subset F_{e,(L,-1)}$.

It is easy to verify from the definitions that $\theta L = -L\theta$. On deduces that θ maps $F_{\sigma(L,\lambda)}$ into $F_{\sigma(L,\lambda')}$ with $P_{\sigma,\lambda'}(X) = \pm P_{\sigma,\lambda}(-X)$.

Similarly θ maps $\tilde{F}_{(L,\lambda)}$ into $\tilde{F}_{(L,\lambda')}$ with $\tilde{P}_{\lambda'}(X) = \pm \tilde{P}_\lambda(-X)$.

Let $x \in F_{e,(L,\lambda)}$ and let n be the smallest integer such that $\theta^{n+1}(x) = 0$, which exists from the fact that θ is nilpotent on F_e . $\theta^n(x) \in \ker(\theta) \subset V_1 \cap W_1 \oplus V_2 \cap W_2 \subset F_{e,(L,1)} \oplus F_{e,(L,-1)}$. As a consequence $\lambda = \pm 1$ and $F_e = F_{e,(L,1)} \oplus F_{e,(L,-1)}$

An analogous argument gives $F_\tau = F_{\tau,(L,0)}$.

Finally let us show that $\lambda = 0, 1, -1 \notin \tilde{\Lambda}$. Suppose the contrary. It exists then an eigenvector x in \tilde{F} associated to the eigenvalue λ . $L(x) = p_{V_1}^{V_2}(x) - p_{W_2}^{W_1}(x) = \lambda x$ implies in the three cases a contradiction with proposition 8.

It follows that $F_e = E_{(L,-1)} \oplus E_{(L,1)}$, as $E_{(L,\lambda)} = F_{e,(L,\lambda)} \oplus F_{\tau,(L,\lambda)} \oplus \tilde{F}_{(L,\lambda)}$.

The same arguments show *mutatis mutandis* that $F_\tau = F_{\tau,(L',1)} \oplus F_{\tau,(L',-1)}$, $V_1 \cap W_2 \subset F_{\tau,(L',1)}$, $V_2 \cap W_1 \subset F_{\tau,(L',-1)}$, and $F_e = F_{e,(L',0)}$.

It follows similarly $F_\tau = E_{(L', -1)} \oplus E_{(L', 1)}$.

The generalized eigenspaces appearing in the proof are invariant by the fact that for any polynomial Q , $Q(L)$ commutes to the action of the representation and so $\ker Q(L)$ (and also $\text{im} Q(L)$) is invariant.

In the reflexive case we have: $L = -L^*$. As a consequence $E_{(L, -1)}$ is orthogonal to any $E_{(L, \lambda)}$ for $\lambda \neq 1$ and $E_{(L, 1)}$ is orthogonal to any $E_{(L, \lambda)}$ for $\lambda \neq -1$. This follows from the relation

$$\langle P_\lambda(L)^{n_\lambda} \cdot, \cdot \rangle = \langle \cdot, P_\lambda(L^*)^{n_\lambda} \cdot \rangle = \langle \cdot, P_\lambda(-L)^{n_\lambda} \cdot \rangle,$$

and from the fact that $P_\lambda(L)^{n_\lambda}$ is an isomorphism of $E_{(L, \mu)}$ for $\mu \neq \lambda$. (kernel lemma)

So $E_{(L, -1)}$, and $E_{(L, 1)}$ are totally isotropic, $E_{(L, -1)} \oplus E_{(L, 1)}$ is orthogonal to all other generalized eigenspaces and non degenerate.

One obtains similarly that $L' = L'^*$. $E_{(L', \lambda)}$ is orthogonal to any $E_{(L', \mu)}$ for $\mu \neq \lambda$. In particular $E_{(L', \lambda)}$ is non degenerate and $E_{(L', -1)}$ is orthogonal to $E_{(L', 1)}$. \square

Let us remark that in the weakly irreducible case, the existence of a decomposition of E into two a direct sum of two degenerate sub-representations implies that $E = F_e$.

Theorem 32. *If E is a weakly irreducible representation preserving the non degenerate reflexive form $\langle \cdot, \cdot \rangle$ and admitting a decomposition into a direct sum of degenerate sub-representations $E = V_1 \oplus V_2$, then $E = E_{(L, 1)} \oplus E_{(L, -1)}$ with $L := p - p^*$. We have: $V_1 \cap V_1^\perp \subset E_{(L, 1)}$ and $V_2 \cap V_2^\perp \subset E_{(L, -1)}$. In addition $E_{(L, 1)}$ et $E_{(L, -1)}$ are totally isotropic and their sum is non degenerate.*

Proposition 33. *If $E = E_1 \oplus E_2$ is a representation preserving the non degenerate reflexive form $\langle \cdot, \cdot \rangle$, and E_1 and E_2 are totally isotropic, then E_2 identifies to E_1^* .*

Proof. As in proposition 30 the map

$$\begin{aligned} E_2 &\rightarrow E_1^* \\ v' &\mapsto (w \mapsto \langle v', w \rangle) \end{aligned}$$

which is injective because $E_2 \cap E_1^\perp = \{0\}$ and surjective for dimension reasons. \square

Lemma 34. *If the representation E admits three sub-representation F_1 , F_2 and F_3 , such that $E = F_1 \oplus F_2 = F_2 \oplus F_3 = F_1 \oplus F_3$, then $E = F_1 \otimes \mathbb{K}^2$ where \mathbb{K}^2 is the trivial representation.*

Proof. Let's note p the projection on F_1 parallelly to F_2 restricted to F_3 . p is an isomorphism of F_3 onto F_1 and commutes with the action of the representation. As a consequence $E = F_1 \oplus F_1 = F_1 \otimes \mathbb{K}^2$. \square

Proposition 35. *If E is a representation admitting two decompositions into supplementary sub-representations $E = V_1 \oplus V_2 = W_1 \oplus W_2$, \tilde{F} identifies to $V \otimes \mathbb{K}^2$ where $V = \tilde{F} \cap V_1$ and \mathbb{K}^2 is the trivial representation.*

Proof. In fact we have $\tilde{F} = \tilde{F} \cap V_1 \oplus \tilde{F} \cap V_2 = \tilde{F} \cap V_1 \oplus \tilde{F} \cap W_1 = \tilde{F} \cap V_2 \oplus \tilde{F} \cap W_1$. We are in the situation described by the preceding lemma. \square

To summarize we have:

Theorem 36. *If E is a representation preserving the non degenerate reflexive form $\langle \cdot, \cdot \rangle$ and the direct sum decomposition $E = V_1 \oplus V_2$, then*

- (i) $E = F_e \oplus^\perp F_\tau \oplus^\perp \tilde{F}$,
- (ii) $F_e = F_e^+ \oplus (F_e^+)^*$ for a totally isotropic representation F_e^+ ,
- (iii) $F_\tau = F_\tau^+ \oplus^\perp F_\tau^-$ for a non degenerate representation F_τ^+ ,
- (iv) $\tilde{F} = \tilde{F}_0 \otimes \mathbb{K}^2$ for a non degenerate representation \tilde{F}_0 and \mathbb{K}^2 being the trivial representation.

4. APPLICATION TO HOLONOMY

A particular case of the preceding is when \mathfrak{g} is a holonomy algebra. We call *formal curvature tensor* an element R of $(E^* \wedge E^*) \otimes E^* \otimes E$ such that for all $x, y, z \in E$ we have: $R(x, y)z + R(y, z)x + R(z, x)y = 0$ (first Bianchi identity). We will suppose that there is a finite set of formal curvature tensors $\{R_1, R_2, \dots, R_m\}$ such that \mathfrak{g} is the linear Lie algebra generated by the $R_i(x, y) \in \text{End}(E)$ for $i = 1 \dots m$ and $x, y \in E$. We will call such an algebra *Berger algebra*. For a holonomy algebra this situation is given by the Ambrose-Singer theorem which relates the curvature tensor of a connected manifold equipped with a torsion-free connection to its holonomy algebra in a point of the manifold. In the following we will write R one of the formal curvature tensors R_1, R_2, \dots, R_m .

Definition 4. *If R is a formal curvature tensor and $\mathfrak{g} \subset \mathfrak{gl}(E)$ a Berger algebra, we say that R matches \mathfrak{g} , if $\forall x, y \in E, R(x, y) \in \mathfrak{g}$.*

4.1. General case.

Lemma 37. *If $\mathfrak{g} \subset \mathfrak{gl}(E)$ is a Berger algebra admitting the invariant spaces F_1, F_2, \dots, F_r with $E = F_1 \oplus F_2 \oplus \dots \oplus F_r$, and if R is a formal curvature tensor which matches \mathfrak{g} , then $\forall i, j, k, k \notin \{i, j\} \Rightarrow \forall x \in F_i, y \in F_j, z \in F_k, R(x, y)z = 0$.*

Proof. Suppose x, y, z as in the statement. Then by the identity

$$R(x, y)z + R(y, z)x + R(z, x)y = 0$$

and by the fact that $R(y, z)x \in F_i$, $R(z, x)y \in F_j$ and $R(x, y)z \in F_k$ it is clear from $(F_i + F_j) \cap F_k = \{0\}$ that $R(x, y)z = 0$. \square

Definition 5. *We will say that the representation $\mathfrak{g} \subset \mathfrak{gl}(E)$ admitting the invariant spaces F_i with $E = F_1 \oplus F_2 \oplus \dots \oplus F_r$ decomposes into an exterior product along the decomposition $E = F_1 \oplus F_2 \oplus \dots \oplus F_r$ if for any $a \in \mathfrak{g}$, $\forall i, a|_{F_i} \in \mathfrak{g}$.*

Proposition 38. *If $\mathfrak{g} \subset \mathfrak{gl}(E)$ is a Berger algebra and preserves V_1, V_2, W_1 and W_2 such that $E = V_1 \oplus V_2 = W_1 \oplus W_2$ then E decomposes into an exterior product along the decomposition $F \oplus \tilde{F}$. If in addition \mathfrak{g} preserves the reflexive form $\langle \cdot, \cdot \rangle$ and if $W_1 = V_1^\perp$ and $W_2 = V_2^\perp$, then E decomposes into an exterior product along the decomposition $F_e \oplus F_\tau \oplus \tilde{F}$.*

Proof. For the first affirmation, this results from the preceding lemma and from the fact that \tilde{F} is of type $\tilde{F}_0 \otimes \mathbb{R}^2 \simeq \tilde{F}_0 \oplus \tilde{F}_0$. In the reflexive case F_e is of type $F_e^+ \oplus (F_e^+)^*$ from which by a similar argument one can deduce the second affirmation. \square

4.2. Metric case. In the metric case the invariant non degenerate reflexive form $\langle \cdot, \cdot \rangle$ is supposed to be bilinear symmetric and $\mathbb{K} = \mathbb{R}$.

It is well known that from the invariance of $\langle \cdot, \cdot \rangle$, the first Bianchi identity and from the antisymmetry in the two first arguments of R , one can deduce

$$\forall x, y, z, t \in E, \langle R(x, y)z, t \rangle = \langle R(z, t)x, y \rangle, (*)$$

for any formal curvature tensor R matching the algebra.

Lemma 39. *If the algebra \mathfrak{g} is Berger, preserves two supplementary spaces V_1 et V_2 and a non degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, and if for $\mathcal{V} = (E, V_1, V_2, V_1^\perp, V_2^\perp)$ $E = F_e$, then one has for any formal curvature tensor R matching \mathfrak{g} and $x, y \in V_1$, $R(x, y) = 0$ and for $x', y' \in V_2$, $R(x', y') = 0$.*

Proof. From the first Bianchi identity one has $\forall z' \in V_2, R(x, y)z' + R(y, z')x + R(z', x)y = 0$. We have: $R(x, y)z' \in V_2$, $R(y, z')x \in V_1$ and $R(z', x)y \in V_1$ by invariance of V_1 and V_2 under the action of $R(x, y) \in \mathfrak{g}$ (respectively $R(y, z') \in \mathfrak{g}$, $R(z', x) \in \mathfrak{g}$). As V_1 and V_2 form a direct sum, one has: $R(x, y)z' = 0$.

Let's show us further $\forall z \in V_1, R(x, y)z = 0$. Let $t' \in V_2$. $\langle R(x, y)z, t' \rangle = -\langle z, R(x, y)t' \rangle = 0$, by the preceding argument. So from $R(x, y)z \in V_1$, it is clear that $R(x, y)z \in V_1 \cap V_2^\perp = \{0\}$ (in F_e).

As a conclusion for $x, y \in V_1, R(x, y) = 0$. Similarly for $x', y' \in V_2, R(x', y') = 0$. \square

Theorem 40. *If the algebra $\mathfrak{g} \subset \mathfrak{gl}(E)$ is Berger, preserves the the two supplementary spaces V_1 and V_2 and a non degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, for $\mathcal{V} = (E, V_1, V_2, V_1^\perp, V_2^\perp)$, one has: $\mathfrak{g}E \subset \ker \theta_{\mathcal{V}}$ and $\mathfrak{gim} \theta_{\mathcal{V}} = \{0\}$.*

Proof. By theorem 36 one has the decomposition into sub-representations $E = (F_e^+ \oplus (F_e^+)^*) \oplus^\perp F_\tau^+ \oplus^\perp F_\tau^- \oplus^\perp (\tilde{F}_0 \otimes \mathbb{R}^2)$ with F_e^+ (and $(F_e^+)^*$) totally isotropic, F_τ^+ , F_τ^- and \tilde{F}_0 non degenerate

For R a formal curvature tensor matching \mathfrak{g} , as $R(x, y) = 0$ for $x \perp y$ (by $(*)$), \mathfrak{g} is generated by the $R(x, y)$ for $(x, y) \in F_e^+ \times (F_e^+)^*$, (respectively $(x, y) \in F_\tau^+ \times F_\tau^+$, resp. $(x, y) \in F_\tau^- \times F_\tau^-$). $R(x, y)$ acts only on $F_e^+ \oplus (F_e^+)^*$ (respectively F_τ^+ , resp. F_τ^-).

For $(x, y) \in F_e^+ \times (F_e^+)^*$, $z \in V_1 \cap F_e$, $t \in V_1 \cap F_e$, one has $\langle R(x, y)z, t \rangle = \langle R(z, t)x, y \rangle = 0$, and similarly for $(x, y) \in F_e^+ \times (F_e^+)^*$, $z \in V_2 \cap F_e$, $t \in V_2 \cap F_e$, one has $\langle R(x, y)z, t \rangle = 0$. So we obtain: $\mathfrak{g}F_e \subset V_1 \cap V_1^\perp \oplus V_2 \cap V_2^\perp \subset \ker(\theta_{\mathcal{V}})$.

Recall that θ maps W_1 into W_2 and W_2 into W_1 .

For $(x, y) \in F_\tau^+ \times F_\tau^+$, $z \in F_\tau^+$, $t \in F_\tau^-$, one has: $\langle \theta(R(x, y)z), t \rangle = \langle R(x, y)z, \theta(t) \rangle = \langle R(z, \theta(t))x, y \rangle = 0$ because $z \perp \theta(t)$. So $\theta(R(x, y)F_\tau^+) \subset F_\tau^- \cap (F_\tau^-)^\perp = \{0\}$. Similarly for $(x, y) \in F_\tau^- \times F_\tau^-$, $\theta(R(x, y)F_\tau^-) = \{0\}$, so $\mathfrak{g}F_\tau \subset \ker(\theta_{\mathcal{V}})$.

$\mathfrak{g}E \subset \ker(\theta_{\mathcal{V}})$ follows from the preceding observations. As θ commutes with every element of \mathfrak{g} , we will have as well: $\mathfrak{gim} \theta = \mathfrak{g}\theta(E) \subset \theta \mathfrak{g}E = \{0\}$. \square

Corollary 41. *Let E be a metric indecomposable representation of the Berger algebra \mathfrak{g} preserving the decomposition $E = V_1 \oplus V_2$ with V_1 or V_2 degenerate. For $\mathcal{V} = (E, V_1, V_2, V_1^\perp, V_2^\perp)$, one has: $\theta_{\mathcal{V}}^2 = 0$.*

Proof. Recall that in the metric indecomposable case with $E = V_1 \oplus V_2$ where V_1 or V_2 is degenerate, one has $E = F_e$. Suppose θ_V^2 is non zero. In this case one can choose a non trivial supplementary space A of $\ker \theta \cap \operatorname{im} \theta$ in $\operatorname{im} \theta$. A is also a supplementary space of $\ker \theta$ in $\ker \theta + \operatorname{im} \theta$. Let us choose a supplementary space B of $\ker \theta + \operatorname{im} \theta$ in E . One has: Because $A \subset \operatorname{im} \theta$, there exists A' subset of E such that $A = \theta A'$. For $a \in \mathfrak{g}$, $aA = a\theta A' = \theta aA' = \{0\}$ by the preceding theorem because $aA' \subset \mathfrak{g}E$. So A is invariant for the action of \mathfrak{g} . $\ker \theta + B$ is a supplementary space of A , which is also invariant by \mathfrak{g} , because $\mathfrak{g}(\ker \theta + B) \subset \ker \theta \subset \ker \theta + B$. So we obtain a new decomposition of E into two \mathfrak{g} -invariant spaces A and $\ker \theta + B$. the action of \mathfrak{g} on A is trivial. So the action of \mathfrak{g} decomposes into an exterior product along the decomposition $A \oplus (\ker \theta + B)$, in contradiction to what we supposed. \square

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